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Div - curl estimates with critical power weights

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1 Introduction

This note is based on [11]. But, the proof of the main result in [11] is not correct, though the result is true. Aim of this note is to give a proof of it, along the talk. It applies Bogovskiĭ formula instead of the Green function for the Neumann problem of Poisson equation, which was used in [11]. I learned this formula from Professor Hideo Kozono, and would like to thank him for teaching me it. It is should be emphasize that thanks to the formula, the result on 3-dimension in [11] is generalized to all dimension.

Div - curl estimate is a inequality of the form, which was firstly studied by Coifman-Linons-Meyer-Semmes [2]: for $p, q \in (n/(n+1), \infty)$ and $1/r = 1/p + 1/q < 1 + 1/n$,

$$\|(u \cdot \nabla)v\|_{H^r} \lesssim \|u\|_{H^q} \|\nabla v\|_{H^p}, \quad (1)$$

where u and v are vector valued functions; $u = \{u_j\}_{j=1}^n, v = \{v_j\}_{j=1}^n$, and $\operatorname{div} u = \nabla \cdot u = 1$. Here

$$(u \cdot \nabla)v = \left(\sum_{j=1}^n u_j \partial_j v_1, \dots, \sum_{j=1}^n u_j \partial_j v_n \right).$$

H^p , ($p \in (0, \infty)$) is the Hardy spaces: for any $\phi \in C_0^\infty(\mathbb{R}^n)$ with $\operatorname{supp} \phi \subset\subset B(0, 1)$ and $\int \phi dx = 0$,

$$\|f\|_{H^p} := \|M_\phi[f]\|_{L^p}, \text{ where } M_\phi[f](x) := \sup_{t>0} |f * \phi_t(x)|,$$

where $g_t(x) := t^{-n}g(x/t)$. In the case $p = \infty$, define $H^\infty := L^\infty$.

As we mentioned in [11], we make use of a real interpolation spaces between weighted Hardy spaces.

Definition 1.1. Let $p, q \in (0, \infty]$ and $\alpha \in \mathbb{R}$. Define Hardy spaces associated with Herz spaces $H_\alpha^{p,q}(\mathbb{R}^n)$ as

$$H_\alpha^{p,q}(\mathbb{R}^n) := \left\{ f \in \mathcal{S}' ; \|f\|_{H_\alpha^{p,q}} := \|M_\phi f\|_{L_\alpha^{p,q}} < \infty \right\},$$

where

$$\|f\|_{L_\alpha^{p,q}} := \left\| \{2^{\alpha k} \|f\|_{L^p(A_k)}\}_{k \in \mathbb{Z}} \right\|_{l_q}.$$

We explain notations. $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$ denote the Schwartz spaces of rapidly decreasing smooth functions and tempered distributions on \mathbb{R}^n , respectively. For a measurable subset $E \subset \mathbb{R}^n$, $|E|$ and χ_E are the volume and the characteristic function of E , respectively. For any integers j , A_j denotes an annulus $\{x \in \mathbb{R}^n; 2^{j-1} \leq |x| < 2^j\}$, and χ_j is the characteristic function of A_j . $B(x, r)$ is a ball in \mathbb{R}^n , centered at x of radius r . $\langle g \rangle_B := |B|^{-1} \int_B f dy$. Also, $A \lesssim B$ means $A \leq cB$ with positive constant c , and $A \approx B$ means $A \lesssim B$ and $B \lesssim A$.

To give $(u \cdot \nabla)v$ a definition as a tempered distribution, we define Y by a space of all locally integrable functions f satisfying that there exist $c_f > 0$ and a seminorm $|\cdot|_S$ of \mathcal{S} so that $\int |f(x)\varphi(x)|dx \leq c_f|\varphi|_S$, for all $\varphi \in \mathcal{S}$.

The main result reads as follows.

Theorem 1.1. *For $n/(n+1) < p < \infty$, it holds*

$$\|(u \cdot \nabla)v\|_{H_{\alpha(p)}^{p,\infty}} \leq c\|u\|_{L^\infty}\|\nabla v\|_{H_{\alpha(p)}^{p,n/(n+1)}},$$

for $u \in L^\infty(\mathbb{R}^n)^n$ with $\operatorname{div} u = 0$ and $v \in (Y \cap W_{loc}^{1,r}(\mathbb{R}^n))^n$ for some $r \in (1, \infty)$, where $\alpha(p) := n(1 - 1/p) + 1$.

Remark 1.1. *The same argument as the proof of Theorem 1.1, we can also show a weak type estimate:*

$$\|(u \cdot \nabla)v\|_{H^{n/(n+1),\infty}} \lesssim \|u\|_{L^\infty}\|\nabla v\|_{H^{n/(n+1)}}, \quad (2)$$

because $\alpha(n/(n+1)) = 0$. Here, $f \in H^{(p,q)} \iff M_\phi[f] \in L^{(p,q)}$, where $L^{(p,q)}$ is the Lorentz spaces. Similar estimates were established by Miyakawa [8]. This can be regarded as an endpoint case with $p = n/(n+1)$ and $q = \infty$ of [2]. The ingredient of the proof of (2) is the pointwise estimate:

$$\|N[v]\|_{L^{n/(n+1),\infty}} \lesssim \|\nabla v\|_{H^{n/(n+1)}}$$

instead of (4). This is achieved from the pointwise estimate (7) and a Fefferman-Stein's vector valued inequality (2) of Theorem 1 in [3].

Motivation of this research comes from the optimal L^2 -energy decay for the incompressible Navier-Stokes equations. Wiegner [12] constructed global weak solutions u having

$$\|u(t)\|_{L^2} \lesssim t^{-(n+2)/4},$$

assuming that initial data $a \in L^2$ satisfying $\|e^{t\Delta}a\|_{L^2} \lesssim t^{-(n+2)/4}$. By Miyakawa-Schonbek [9], it is well-known that the decay order $(n+2)/4$ is optimal. $H_{\alpha(p)}^p$ is relevant to this order $(n+2)/4$, because one has that for $p \in (0, 2]$,

$$\|e^{t\Delta}a\|_{L^2} \lesssim t^{-(n+2)/4}\|a\|_{H_{\alpha(p)}^p},$$

see [10] for the proof. The present author [10] investigated the L^2 decay of mild solutions by Kato [5] and constructed solutions whose decay order of L^2 energy is $\gamma < (n+2)/4$. One of reasons why the order γ in [10] did not reach to the optimal order $(n+2)/4$ is that div-curl estimate in [10] cannot allow us to deal with the critical exponent $\alpha = \alpha(p)$. As mentioned in Remark 7.3 in [7], the bilinear term $(u \cdot \nabla)v$ does not belong to $H_{\alpha(p)}^p$. This observation tells us that if we try to establish div-curl estimate with $\alpha = \alpha(p)$, we have to replace $H_{\alpha(p)}^p$ in the left hand side by some larger spaces. For the purpose, we use Hardy spaces associated to Herz spaces, as in [6] and [7]. Although, a critical div - curl lemma is proved in this article, the author does not know whether or not it is possible to construct global solutions having optimal L^2 decay from the similar argument as the previous paper [10].

2 Proof

Before we start the proof of Theorem 1.1, we point out the mistake in [?]. In [?], the Green function for the Neumann problem of Poisson equation:

$$-\Delta h = g \text{ in } B, \quad g = 0 \text{ on } \partial B$$

for $g \in C_0^\infty(B)$ with $\int g dx = 0$, is applied. I treated the solution h as a $C_0^2(B)$ function in [11]. Although $h \in C^2(\mathbb{R}^n)$, this is not true in general. To overcome this difficulty, as we mentioned above, we apply Bogovskiĭ formula. This is a representation, with a kernel function, of solutions to the divergence equation.

2.1 Bogovskiĭ formula

Let B be a ball in \mathbb{R}^n and $g \in C_0^\infty(B)$ with $\int g dx = 0$. We refer Lemma III.3.1 in [4] for next lemma.

Lemma 2.1. *There exists a vector function $\mathbf{K} = \{K_j\}_{j=1}^n$ on $\mathbb{R}^n \times \mathbb{R}^n \setminus \{(x, y) : x = y\}$ so that $\mathbf{G}_B(x) := \int \mathbf{K}(x, y)g(y)dy \in C_0^\infty(B)^n$ is a solution to the divergence equation $\nabla \cdot \mathbf{G}_B = g$ on B satisfying that for $q \in (1, \infty)$*

$$\|\mathbf{G}_B\|_{L^q} \lesssim |B|^{1/n} \|g\|_{L^q} \quad \text{and} \quad \|\nabla \cdot \mathbf{G}_B\|_{BMO} \lesssim \|g\|_{L^\infty}.$$

Remark 2.1. *The $L^\infty - BMO$ estimate above is deduced from the fact that the operator $g \mapsto \mathbf{G}_B$ is a Calderón-Zygmund operator.*

2.2 Vector valued restricted weak type inequality

Another ingredient for the proof of Theorem 1.1 is a vector-valued “restricted weak” type inequality for Hardy-Littlewood maximal operator; for $r \in (0, \infty)$

$$M_r f(x) := \sup_{B \ni x} \langle |f|^r \rangle_B^{1/r},$$

where the supremum is taken over all ball B containing x . Define $Mf(x) := M_1 f(x)$. The following is a generalization of the result of Fefferman-Stein [3], and the proof is found in [11].

Proposition 2.1. *For $1 < r, p < \infty$ and $\alpha = n(1 - 1/p)$,*

$$\left\| \left(\sum_{l=1}^{\infty} (Mf_l)^r \right)^{1/r} \right\|_{L_{\alpha}^{p, \infty}} \lesssim \left\| \left(\sum_{l=1}^{\infty} |f_l|^r \right)^{1/r} \right\|_{L_{\alpha}^{p, 1}}.$$

This can be rewritten as the following form.

Corollary 2.1. *For $0 < r < 1 < p < \infty$ and $\alpha = n(1/r - 1/p)$,*

$$\left\| \sum_{l=1}^{\infty} M_r f_l \right\|_{L_{\alpha}^{p, \infty}} \lesssim \left\| \sum_{l=1}^{\infty} |f_l| \right\|_{L_{\alpha}^{p, r}}.$$

2.3 Complete of the proof of Theorem 1.1

The proof is almost same as that in [11] except for applying Lemma 2.1 instead of the Green function for the Neumann problem of Poisson equations.

Because

$$\|(u \cdot \nabla)v\|_{H_{\alpha(p)}^{p, \infty}} = \sum_{k=1}^n \left\| \sum_{j=1}^n u_j \partial_j v_k \right\|_{H_{\alpha(p)}^{p, \infty}} = \sum_{k=1}^n \left\| M_{\phi} \left(\sum_{j=1}^n u_j \partial_j v_k \right) \right\|_{L_{\alpha(p)}^{p, \infty}},$$

it is enough to show the inequality

$$\left\| M_{\phi} \left(\sum_{j=1}^n u_j \partial_j v \right) \right\|_{L_{\alpha(p)}^{p, \infty}} \lesssim \|u\|_{L^{\infty}} \|\nabla v\|_{H_{\alpha(p)}^{p, n/(n+1)}},$$

for all divergence free vector fields u and functions $v \in Y \cap W_{loc}^{1, r}$. Firstly, we give a definition of $\sum_{j=1}^n u_j \partial_j v$ as a tempered distribution as follows; for $\varphi \in \mathcal{S}$

$$\left\langle \sum_{j=1}^n u_j \partial_j v, \varphi \right\rangle := - \sum_{j=1}^n \int u_j(y) v(y) \partial_j \varphi(y) dy.$$

Our assumption ensures that the integral in the right hand side absolutely converges. Then, it follows

$$\sum_{j=1}^n u_j \partial_j v * \phi_t(x) = -C_\phi \|u\|_{L^\infty} \int v(y) \left[\sum_{j=1}^n \tilde{u}_j(y) \partial_{y_j} \phi_t(x-y) \right] dy,$$

where C_ϕ is a constant depending on ϕ , and $\tilde{u}_j(y) = \frac{u_j(y)}{C_\phi \|u\|_{L^\infty}}$. Owing to the divergence free condition on u , we see that for every $x \in \mathbb{R}^n$

$$\sum_{j=1}^n \tilde{u}_j(y) \partial_{y_j} \phi_t(x-y) = \sum_{j=1}^n \partial_{y_j} (\tilde{u}_j(y) \phi_t(x-y)) \quad \text{in } \mathcal{S}'(\mathbb{R}_y^n). \quad (3)$$

Hence, we obtain the pointwise estimate

$$M_\phi \left(\sum_{j=1}^n u_j \partial_j v \right) (x) \leq C_\phi \|u\|_{L^\infty} N[v](x),$$

where

$$N[v](x) := \sup_{t>0} \left| \int v(y) g(y) dy \right| \quad \text{and } g(y) = g(y; x, t) := \sum_{j=1}^n \tilde{u}_j(y) \partial_{y_j} \phi_t(x-y).$$

It is enough to prove that

$$\|N[v]\|_{L_{\alpha(p)}^{p,\infty}} \lesssim \|\nabla v\|_{H_{\alpha(p)}^{p,n/(n+1)}}. \quad (4)$$

To show this from a pointwise estimate, we make use of Bogovskii formula Lemma 2.1 and the atomic decomposition in $H_{\alpha(p)}^{p,n/(n+1)}$ due to Miyachi [7].

Fix $x \in \mathbb{R}^n$ and $t \in (0, \infty)$. There exists $\varepsilon_0 > 0$ so that for all $\varepsilon \in (0, \varepsilon_0)$, $\text{supp } g * \phi_\varepsilon \subset \subset B(x, t)$. Take $\eta_0 \in C_0^\infty(\mathbb{R}^n)$ such that $\eta(y) = \eta(y; x, t) := \eta_0\left(\frac{y-x}{t}\right)$ satisfies that $0 \leq \eta \leq 1$ and for $\varepsilon \in (0, \varepsilon_0)$

$$\text{supp } g * \phi_\varepsilon \subset \subset \text{supp } \eta \subset \subset B(x, t), \quad \text{and } \eta \equiv 1 \text{ on } \text{supp } g * \phi_\varepsilon$$

Remark that $\|\eta\|_{L^p} = ct^{n/p}$ for all $p \in [1, \infty]$ with c independent of x, t and ε . Next we see that

$$\sigma_\varepsilon := \|\eta\|_{L^1}^{-1} \int g * \phi_\varepsilon dy \rightarrow 0 \text{ as } \varepsilon \searrow 0.$$

In fact, for a test function $\rho \in C_0^\infty(B(x, 2t))$ with $\rho \equiv 1$ on $B(x, t)$, we have from (3)

$$\|\eta\|_{L^1} \sigma_\varepsilon = \langle g, \rho * \phi_\varepsilon \rangle \rightarrow \langle g, \rho \rangle = 0.$$

For $\varepsilon \in (0, \varepsilon_0)$, letting $g^\varepsilon := g * \phi_\varepsilon - \sigma_\varepsilon \eta \in C_0^\infty(B(x, t))$, we obtain $\int g^\varepsilon dy = 0$. From Lemma 2.1,

$$\mathbf{G}^\varepsilon(y) := \int \mathbf{K}(y, z) g^\varepsilon(z) dz \in C_0^\infty(B(x, t))$$

solves the Dirichlet problem of Poisson equation:

$$\nabla \cdot \mathbf{G}^\varepsilon = g^\varepsilon \text{ in } B(x, t), \quad \mathbf{G}^\varepsilon = 0 \text{ on } \partial B(x, t).$$

Further, \mathbf{G}^ε fulfills the following estimates: for all $q \in (1, \infty)$,

$$\|\mathbf{G}^\varepsilon\|_{L^q} \lesssim t^{-n+n/q} \quad \text{and} \quad \|\partial_j \mathbf{G}^\varepsilon\|_{BMO} \lesssim t^{-(n+1)}. \quad (5)$$

Indeed, these follows that $\|g^\varepsilon\|_{L^q} \lesssim t^{-1-n+n/q}$ and $\|g^\varepsilon\|_{L^\infty} \lesssim t^{-(n+1)}$, respectively.

Integration by parts yields

$$\int v g dy = - \lim_{\varepsilon \rightarrow 0} \int \nabla v \cdot \mathbf{G}^\varepsilon dy.$$

Therefore, one obtains

$$\left| \int v g dy \right| \leq \sum_{k=1}^n \limsup_{0 < \varepsilon < \varepsilon_0} \left| \int \partial_k v \mathbf{G}_k^\varepsilon dy \right|.$$

Since $\partial_k v \in H_{\alpha(p)}^{p, n/(n+1)}$, following Miyachi [7], it can be decomposed as

$$\partial_k v = \sum_{j=1}^{\infty} a_j^{(k)}$$

where $\text{supp } a_j^{(k)} \subset B_j = B(x_j, r_j)$, $a_j^{(k)} \in L^\infty$ and $\int x^\beta a_j^{(k)}(x) dx = 0$ for $|\beta| \leq 1$, also

$$\left\| \sum_{j=1}^{\infty} \|a_j^{(k)}\|_{L^\infty} \chi_{B_j} \right\|_{L_{\alpha(p)}^{p, n/(n+1)}} \lesssim \|\partial_k v\|_{H_{\alpha(p)}^{p, n/(n+1)}}.$$

and one obtains

$$\left| \int v g dy \right| \leq \sum_{k=1}^n \sum_{j=1}^{\infty} \limsup_{0 < \varepsilon < \varepsilon_0} \left| \int a_j^{(k)} \mathbf{G}_k^\varepsilon dy \right|.$$

From (5), we immediately see that

$$\left| \int a_j^{(k)} \mathbf{G}_k^\varepsilon dy \right| \leq \|a_j^{(k)}\|_{L^\infty} |B(x, t)|^{1-1/q} \|\mathbf{G}_k^\varepsilon\|_{L^q} \lesssim \|a_j^{(k)}\|_{L^\infty}.$$

When $x \notin 4B_j$, if $Ct < |x - x_j|$ with $C > 8/3$, then it holds $B_j \cap B(x, t) = \emptyset$ and $\int a_j^{(k)} \mathbf{G}_k^\varepsilon dy = 0$. On the other hand, if $Ct \geq |x - x_j|$, then we can derive the decay estimate

$$\limsup_{0 < \varepsilon < \varepsilon_0} \left| \int a_j^{(k)} \mathbf{G}_k^\varepsilon dy \right| \lesssim \|a_j^{(k)}\|_{L^\infty} \left(\frac{r_j}{|x - x_j|} \right)^{n+1}. \quad (6)$$

We may assume $x \neq x_j$. Using the moment condition on $a_j^{(k)}$ twice, one has

$$\begin{aligned} \int a_j^{(k)}(y) \mathbf{G}_k^\varepsilon(y) dy &= \int a_j^{(k)}(y) (\mathbf{G}_k^\varepsilon(y) - \mathbf{G}_k^\varepsilon(x_j)) dy \\ &= \sum_{s=1}^n \int_0^1 \int a_j^{(k)}(y) (y - x_j)_s (\partial_s \mathbf{G}_k^\varepsilon)(\theta y + (1 - \theta)x_j) dy d\theta \\ &= \sum_{s=1}^n \int_0^1 \int a_j^{(k)}(y) (y - x_j)_s [(\partial_s \mathbf{G}_k^\varepsilon)(\theta y + (1 - \theta)x_j) - \langle \partial_s \mathbf{G}_k^\varepsilon \rangle_{B(x_j, \theta r_j)}] dy d\theta. \end{aligned}$$

From this representation, the decay estimate (6) is derived as follows from (5);

$$\begin{aligned} \left| \int a_j^{(k)}(y) \mathbf{G}_k^\varepsilon(y) dy \right| &\lesssim r_j \|a_j^{(k)}\|_{L^\infty} \sum_{s=1}^n \int_0^1 \theta^{-n} \int_{B(x_j, \theta r_j)} |\partial_s \mathbf{G}_k^\varepsilon(y) - \langle \partial_s \mathbf{G}_k^\varepsilon \rangle_{B(x_j, \theta r_j)}| dy d\theta \\ &\lesssim r_j^{n+1} \|a_j^{(k)}\|_{L^\infty} \sum_{s=1}^n \|\partial_s \mathbf{G}_k^\varepsilon\|_{BMO} \\ &\lesssim \left(\frac{r_j}{t} \right)^{n+1} \|a_j^{(k)}\|_{L^\infty} \\ &\lesssim \left(\frac{r_j}{|x - x_j|} \right)^{n+1} \|a_j^{(k)}\|_{L^\infty}. \end{aligned}$$

As mentioned in [7], because $\left(\frac{1}{1 + |x - x_j|/r_j} \right)^{n+1} \approx M_{n/(n+1)}(\chi_{B_j})(x)$, as a consequence it follows that for all $x \in \mathbb{R}^n$,

$$N[v](x) = \sup_{t>0} \left| \int v(y) g(y; x, t) dy \right| \lesssim \sum_{k=1}^n \sum_{j=1}^\infty \|a_j^{(k)}\|_{L^\infty} M_{n/(n+1)}(\chi_{B_j})(x). \quad (7)$$

Now, we apply Corollary 2.1 with $r = n/(n+1)$ and obtain

$$\|N[v]\|_{L_{\alpha(p)}^{p,\infty}} \lesssim \sum_{k=1}^n \left\| \sum_{j=1}^\infty \|a_j^{(k)}\|_{L^\infty} \chi_{B_j} \right\|_{L_{\alpha(p)}^{p,n/(n+1)}} \lesssim \sum_{k=1}^n \|\partial_k v\|_{H_{\alpha(p)}^{p,n/(n+1)}} = \|\nabla v\|_{H_{\alpha(p)}^{p,n/(n+1)}}.$$

Here we have used $n(1 - 1/p) + n((n+1)/n - 1) = n(1 - 1/p) + 1 = \alpha(p)$. The proof is completed.

Remark 2.2. In [1], the pointwise estimate (6) with $n+1-\varepsilon$ replaced by $n+1$ was proved.

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